IPMU – March 11th, 2025

Cohomological integrality for symmetric quotient stacks

Outline

X smooth affine G-variety

G reductive algebraic group

$/\mathbb{C}$.

Study $\underset{\text{quotient stack}}{X/G} \longrightarrow \underset{\text{GIT quotient}}{X /\!\!/ G} := \mathrm{Spec}(\mathbb{C}[X]^G)$

Interplay between $\underset{\text{equivariant cohomology}}{H^*_G(X)}$ and $\underset{\text{intersection cohomology}}{IH^*(X/\!\!/G)}$

e.g. if $X = V$ vector space, $H^*_G(V) \cong$ polynomial ring

while $IH^*(V/\!\!/G)$ hard to compute

also: $V/G \to V/\!\!/G$ is a local model for good moduli spaces of smooth stacks $\mathfrak{X} \longrightarrow X$ [étale slices]

1 - Situation

$G = GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$, $(\mathbb{C}^*)^N$, $Sp_{2n}(\mathbb{C})$, ...

More generally, G: reductive group (unipotent radical is trivial)

$\underset{\text{char } 0}{=}$ linearly reductive

(finite-dimensional representations are semisimple)

<u>non-example</u>: $G = \mathbb{G}_a$ additive group

acts on $V = \mathbb{C}^2$ via $\mathbb{G}_a \hookrightarrow \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$.

V is non-trivial extension of $\mathbb{C}$ by $\mathbb{C}$.

* $T \subset G$ maximal torus. $T \cong (\mathbb{C}^*)^{\mathrm{rank}(G)}$

e.g. diag $\simeq (\mathbb{C}^*)^n \subset GL_n(\mathbb{C})$.

* representation: $G \longrightarrow GL(V)$, V $\mathbb{C}$ vector space, finite-dimensional.

$GL_2(\mathbb{C}), SL_2(\mathbb{C}) \curvearrowright \mathbb{C}^2$.

characters: $X^*(T) = \{\alpha : T \to \mathbb{G}_m\} \cong \mathbb{Z}^{\mathrm{rk}\,G}$

cocharacters $X_*(T) = \{\lambda : \mathbb{G}_m \to T\} \cong \mathbb{Z}^{\mathrm{rk}\,G}$

Pairing $\alpha \circ \lambda : \mathbb{G}_m \longrightarrow \mathbb{G}_m$

$$z \longmapsto z^{\langle \lambda, \alpha \rangle}$$

$$\langle -,- \rangle : X_*(T) \times X^*(T) \to \mathbb{Z}.$$

Weights $T \curvearrowright V$ diagonalizable:

$$V = \bigoplus_{\alpha \in X^*(T)} V_\alpha$$

$$V_\alpha = \{ v \in V \mid t \cdot v = \alpha(t) \cdot v \quad \forall t \in T \}$$

$$\mathcal{W}(V) = \{ \alpha \in X^*(T) \mid V_\alpha \neq 0 \} \text{ weights of } V.$$

In particular, $\mathcal{W}(\mathfrak{g})$ weights of $\mathfrak{g} = \mathrm{Lie}(G)$.

ex. $GL_2(\mathbb{C}) \curvearrowright \mathbb{C}^2 = \underset{(1,0)}{\mathbb{C}e_1} \oplus \underset{(0,1)}{\mathbb{C}e_2}$

$\quad \cup$

$(\mathbb{C}^*)^2$

$$(t_1, t_2) e_1 = t_1 e_1$$
$$(t_1, t_2) e_2 = t_2 e_2$$

V symmetric: $\dim V_\alpha = \dim V_{-\alpha} \quad \forall \alpha \in X^*(T)$

$\Leftrightarrow$ $V \cong V^*$ (a representation is determined by its character)

sort of weakening of symplecticity, appears sometimes when def Coulomb branches.

ex: $\cdot\, T^*V = V \oplus V^*$, V rep of G

$\cdot$ any V rep of $SL_2(\mathbb{C})$

$\cdot$ $\mathfrak{g}$ adjoint of G

$\cdot$ any representations in type B_n, C_n, E_7, E_8, G_2

$O(2n+1)$ $\quad$ $Sp(2n)$

More generally: V weakly symmetric if $\mathcal{W}(V) \bmod \mathbb{Q}_+^* \subset X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q} / \mathbb{Q}_+^*$ is symmetric.

Question: rep. theoretic interpretation?

Weyl group $\quad W = N_G(T)/T$

$$N_G(T) = \{ g \in G \mid gTg^{-1} = T \}$$

$$W_{GL_n} \cong W_{SL_n} \cong \mathfrak{S}_n \text{ symmetric group.}$$

In general: W is a Coxeter group.

T torus: $W_T = \{e\}$

$W \curvearrowright$ weights of V $= \mathcal{W}(V)$.

Cohomological integrality

$H_G^*(V)$ equivariant cohomology

V v.space $\Rightarrow$ contractible

$H_G^*(V) \cong H_G^*(pt) \cong H^*(BG)$

E_G contractible space with free G action.

$BG = EG/_G$.

ex: $H^*_{\mathbb{C}^*}(pt)$

$G = \mathbb{C}^* \curvearrowright \mathbb{C}^N \setminus \{0\}$ free

$$\mathbb{C}^\infty \setminus \{0\} = \bigcup_N \mathbb{C}^N \setminus \{0\}$$

$$\mathbb{P}^\infty = \mathbb{C}^\infty \setminus \{0\} / \mathbb{C}^*$$

$$H^*(\mathbb{P}^N) \cong \mathbb{Q}[x] / (x^{N+1}) \qquad \deg(x) = 2$$

$$H^*(\mathbb{P}^\infty) = H^*_{\mathbb{C}^*}(pt) \cong \mathbb{Q}[x]$$

$G = T = (\mathbb{C}^*)^n$ $\quad H_T^*(pt) \cong \mathbb{Q}[x_1, \dots, x_n]$

G general $\quad H_G^*(pt) \cong H_T^*(pt)^W \qquad T \subset G$ max torus.

$\rightsquigarrow H_G^*(pt) \cong \mathbb{Q}[x_1, \dots, x_n]^{\mathfrak{S}_n}$ symmetric polynomials

$G = GL_2(\mathbb{C}) \qquad \mathbb{Q}[x_1 + x_2, x_1 x_2]$

In general $H^*_G(pt)$ is a polynomial algebra
in particular, $\dim_{\mathbb{Q}} H^*_G(pt) = +\infty$.

Cohomological integrality

Extract a finite-dimensional subspace
$$P_0 \subset H^*(V/G)$$
that generates (in the sense of parabolic induction)

P_0 = "cuspidal cohomology" of V/G.
↳ analogy with { character sheaves (rep of fin grps of Lie type)
{ Hecke eigensheaves (Langlands)

2- Context and motivation

ⓐ Topology of the action of G on V (= of the quotient stack V/G)
of the GIT quotient $V /\!/ G \overset{def}{=} \mathrm{Spec}(\mathbb{C}[V]^G)$
finite type affine scheme (Hilbert)

$V/\!/G$ classifies closed G-orbits in V.

ex: ① $\mathbb{C}^* \curvearrowright \mathbb{C}^N$ points 1 $\quad \mathbb{C}^N /\!/ \mathbb{C}^* = pt$
② $\mathbb{C}^* \curvearrowright \mathbb{C}^2$ $\quad t \cdot (u,v) = (tu, t^{-1}v)$
$\lambda \neq 0$ $\{xy = \lambda\}$ are the closed orbits
$\{0\}$

$\rightsquigarrow \mathbb{C}^2 /\!/ \mathbb{C}^* \cong \mathbb{C}$
$\mathbb{C}[x,y]^{\mathbb{C}^*} \cong \mathbb{C}[xy] \subset \mathbb{C}[x,y]$.

③ $G \curvearrowright \mathfrak{g}$ adjoint rep.

$\mathfrak{g}/\!/G \cong \mathfrak{t}/\!/W$ $\qquad \mathfrak{t} = \mathrm{Lie}\, T$

$\cong \mathbb{A}^{\mathrm{rk}\, G}$

④ non smooth:

$\mathbb{C}^* \curvearrowright \mathbb{C}^4 \quad t\cdot(u,v,w,x) = (tu, tv, t^{-1}w, t^{-1}x)$

$\mathbb{C}^4/\!/\mathbb{C}^* \cong \mathrm{Spec}\left(\mathbb{C}[ac, ad, bc, bd]\right)$

$\cong \mathrm{Spec}\left(\mathbb{C}[A,B,C,D]/\langle AD = BC\rangle\right)$

Cohomological integrality $\rightsquigarrow$ algorithmic computation of

$IH^*(V/\!/G)$ [Bu - Davison - Ibanez Nunez - Kinjo - Padurariu]

$IH(X) = \left\{\begin{array}{l} \text{intersection cohomology} = \left\{\begin{array}{l}\text{singular cohomology of } X \text{ smooth} \\ \text{encodes information regarding singularities otherwise.}\end{array}\right. \end{array}\right.$

③ Operations

Parabolic induction

V representation of G

$\lambda : \mathbb{G}_m \longrightarrow T$ cocharacter

$$G^{\lambda} = \{ g \in G \mid \lambda(t) g \lambda(t)^{-1} = g \quad \forall t \in \mathbb{C}^* \}$$

$\subset G$ Levi subgroup

<u>Note</u> G^{λ} reductive

$T \subset G^{\lambda}$.

$$V^{\lambda} = \{ v \in V \mid \lambda(t) v \lambda(t)^{-1} = v \quad \forall t \in \mathbb{C}^* \}$$

$\subset V$ subspace

$$G^{\lambda \geq 0} = \{ g \in G \mid \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists} \}$$

$\subset G$ parabolic subgroup

$$V^{\lambda \geq 0} = \{ v \in V \mid \lim_{t \to 0} \lambda(t) \cdot v \text{ exists} \}$$

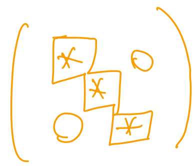
$\subset V$

subspace

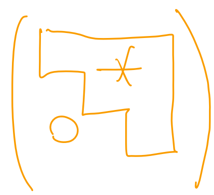
$G = GL_n \qquad V = T^* \mathbb{C}^n$

$\mathbb{G}_m \longrightarrow GL_n$

$t \longmapsto \begin{pmatrix} t^2 & & 0 \\ & t & \\ 0 & & 1 \end{pmatrix}$



$V^{\lambda} = T^* \begin{pmatrix} 0 \\ 0 \\ * \end{pmatrix}$



$V^{\lambda \geq 0} = \mathbb{C}^n \oplus \begin{pmatrix} 0 \\ 0 \\ * \end{pmatrix}$

Induction diagram

$$\begin{tikzcd} & V^{\lambda \geq 0}/G^{\lambda \geq 0} \arrow[dl, "q_\lambda"', "\text{smooth}"] \arrow[dr, "p_\lambda", "\text{proper}"'] & \\ V^\lambda / G^\lambda & & V/G \end{tikzcd}$$

$$\mathrm{Ind}_\lambda := p_\lambda{}^* q_\lambda{}^* : H^*(V^\lambda / G^\lambda) \longrightarrow H^*(V/G)$$

parabolic induction

$$\mathrm{Ind}_\lambda : \mathbb{Q}[x_1, \ldots, x_r]^{W^\lambda} \longrightarrow \mathbb{Q}[x_1, \ldots, x_r]^W.$$

$\exists$ translation of coh degree making Ind_λ graded.

Explicit formula:

$\alpha \in X^*(T)$ may be seen as an element of $H_T^*(pt) \cong \mathrm{Sym}(\mathfrak{t}^*)$
$\alpha : T \to \mathbb{G}_m \quad d\alpha(1) : \mathfrak{t} \to \mathbb{C}$.
$\in \mathfrak{t}^*$

$$k_\lambda := \frac{\displaystyle\prod_{\substack{\alpha \in \mathcal{W}(V) \\ \langle \lambda, \alpha \rangle < 0}} \alpha^{\dim V_\alpha}}{\displaystyle\prod_{\substack{\alpha \in \mathcal{W}(\mathfrak{g}) \\ \langle \lambda, \alpha \rangle < 0}} \alpha^{\dim \mathfrak{g}_\alpha}} \in \mathrm{Frac}\left(H_T^*(pt)\right)$$

$$\mathrm{Ind}_\lambda(f) = \frac{1}{|W^\lambda|} \sum_{w \in W} w \cdot (f k_\lambda).$$

Proof: Calculation after localization and computation of Euler class, using Borel-Weil-Bott thm.

Refinement of this approach

$$\begin{tikzcd}
 & V^{\lambda \geq 0}/G^{\lambda \geq 0} \arrow[dl, "q_\lambda"'] \arrow[dr, "p_\lambda"] & \\
V^\lambda/G^\lambda \arrow[d, "\pi_\lambda"'] & \circlearrowright & V/G \arrow[d, "\pi"] \\
V^\lambda /\!\!/ G^\lambda \arrow[rr, "\iota_\lambda"'] & & V /\!\!/ G
\end{tikzcd}$$

finite morphism

$$\mathrm{Ind}_\lambda : \iota_{\lambda *} \pi_{\lambda *} \mathbb{Q}_{V^\lambda/G^\lambda} \longrightarrow \pi_* \mathbb{Q}_{V/G}$$

Cohomological integrality theorem

$$\lambda \sim \mu \iff \begin{cases} G^\lambda = G^\mu \\ V^\lambda = V^\mu \end{cases}$$

$$\rightsquigarrow \mathcal{P}_V = X_*(T)/\sim \quad \text{finite set}$$

$$\circlearrowleft W$$

$$G_\lambda = \ker\left(G^\lambda \to GL(V^\lambda)\right) \cap Z(G^\lambda) \subset G \quad \text{normal subgroup}$$

$$W_\lambda = \{ w \in W \mid w \cdot \lambda \sim \lambda \} \subset W \quad \text{subgroup}$$

$$\varepsilon_{V,\lambda} : W_\lambda \longrightarrow \{\pm 1\} \quad \text{such that}$$

$$k_{w.\lambda} = \varepsilon_{V,\lambda}(w)\, k_\lambda \quad \forall w \in W_\lambda .$$

Theorem (H-2024) Let V be a weakly symmetric representation of a reductive group G.
There exists bounded cohomologically graded complexes of monodromic mixed Hodge modules $\mathcal{BPS}_{V/G,\lambda}$, which W_λ-equivariant such that the morphism

$$\bigoplus_{\tilde{\lambda} \in \mathcal{P}_V/W} \left[i_{\lambda *}\mathcal{BPS}_{V/G,\lambda} \otimes H^*(pt/G_\lambda) \right]^{\varepsilon_{V,\lambda}} \xrightarrow{\mathrm{Ind}_\lambda} \pi_* \mathbb{Q}^{vir}_{V/G}$$

is an isomorphism in $\mathcal{D}^+(\mathrm{MMHM}(V/\!/G))$.

Conjecture - Theorem when V is orthogonal
[Bu-Davison-Ibáñez Núñez-Kinjo-Pădurariu]

$$\mathcal{BPS}_{V/G,\lambda} \simeq \begin{cases} \mathcal{IC}(V^\lambda/\!/G^\lambda) \otimes \mathcal{L}^{\dim G_\lambda/2} & \text{if } \dim V^\lambda/G^\lambda = \dim V^\lambda/\!/G^\lambda - \dim G_\lambda \\ 0 & \text{otherwise} \end{cases}$$

Direct calculations when $G = \mathbb{C}^*$.
"cuspidal cohomology is intersection cohomology".

(1) $\mathbb{C}^* \curvearrowright V = \bigoplus_{k \in \mathbb{Z}} V_k$

For simplicity, we assume $V_0 = 0$.

$$\lambda_0 : \mathbb{C}^* \longrightarrow \mathbb{C}^*$$
$$t \longmapsto 1$$

$$\lambda_1 : \mathbb{C}^* \longrightarrow \mathbb{C}^*$$
$$t \longmapsto t$$

$\mathcal{P}_V = \{\bar{\lambda}_0, \bar{\lambda}_1\}$; no Weyl group

$V^{\lambda_0} = V$, $G^{\lambda_0} = G$, $G_{\lambda_0} = \{1\}$, $k_{\lambda_0} = 1$

$V^{\lambda_1} = pt$, $G^{\lambda_1} = G$, $G_{\lambda_1} = G$, $k_{\lambda_1} = \prod_{k>0} (kx)^{\dim V_k} \in \mathbb{Q}[x]$.

$$\mathrm{Ind}_{\lambda_1, \lambda_0} : \mathbb{Q}[x] \longrightarrow \mathbb{Q}[x]$$
$$f(x) \longmapsto k_{\lambda_1} \cdot f(x)$$
$$k_{\lambda_1} = C_V \cdot x^{\sum_{k>0} \dim V_k}.$$

$P_{\lambda_0} = \mathbb{Q}[x]_{\deg < \sum_{k>0} \dim V_k}$

$P_{\lambda_1} = \mathbb{Q}$.

Integrality isomorphism

$$\begin{aligned} P_{d_0} \oplus \left(P_{d_1} \otimes \mathbb{Q}[x]\right) &\longrightarrow \mathbb{Q}[x] \\ (f,g) &\longmapsto f + k_{d_1} \cdot g. \end{aligned}$$

clearly an isomorphism

5- Examples

(1) $\overset{G}{\overset{\shortparallel}{GL_2(\mathbb{C})}} \curvearrowright \overset{V}{\overset{\shortparallel}{(T^*\mathbb{C}^2)^g}} \quad g \geq 0 \qquad T = (\mathbb{C}^*)^2 \subset GL_2(\mathbb{C})$

$$\lambda_0 : \mathbb{G}_m \longrightarrow T$$
$$t \longmapsto 1$$

$$\lambda_1 : \mathbb{G}_m \longrightarrow T$$
$$t \longmapsto (t,1)$$

$$\lambda_2 : \mathbb{G}_m \longrightarrow T$$
$$t \longmapsto (t,t^2)$$

$$\lambda_3 : \mathbb{G}_m \longrightarrow T$$
$$t \longmapsto (t,t)$$

- $V^{\lambda_0} = V$, $G^{\lambda_0} = G$, $G_{\lambda_0} = \{1\}$, $W_{\lambda_0} = W$, $k_{\lambda_0} = 1$, $\varepsilon_{V,\lambda_0} = \mathrm{triv}$
- $V^{\lambda_1} = (T^*(0 \oplus \mathbb{C}))^g$, $G^{\lambda_1} = T$, $G_{\lambda_1} = \mathbb{C}^* \times \{1\}$, $W_{\lambda_1} = \{1\}$, $\varepsilon_{V,\lambda_1} = \mathrm{triv}$ $\qquad k_{\lambda_1} = \dfrac{x_1{}^g}{x_1 - x_2}$

$V^{\lambda_2} = \{0\}$, $G^{\lambda_2} = T$, $G_{\lambda_2} = T$, $W_{\lambda_2} = W$

$$k_{\lambda_2} = \frac{(x_1x_2)^g}{x_1 - x_2} \qquad \varepsilon_{V,\lambda_2} = \mathrm{sgn}$$

- $V^{\lambda_3} = \{0\}$, $G^{\lambda_3} = G$, $G_{\lambda_3} = G$, $W_{\lambda_3} = W$,
  $k_{\lambda_3} = (x_1 x_2)^g$, $\varepsilon_{V,\lambda_3} = \mathrm{sgn}$.

Some calculations:

$$P_{\lambda_0} = \bigoplus_{j=0}^{g-2} \mathbb{Q}(x_1+x_2)^j \subset H^*(V/G) \cong \mathbb{Q}[x_1+x_2, x_1x_2]$$

$$P_{\lambda_1} = \bigoplus_{j=0}^{g-1} \mathbb{Q}x_2^j \subset H^*(V^{\lambda_1}/G^{\lambda_1}) \cong \mathbb{Q}[x_1, x_2]$$

$$P_{\lambda_2} = \mathbb{Q} \subset H^*(V^{\lambda_2}/G^{\lambda_2}) \cong \mathbb{Q}[x_1, x_2]$$

$$P_{\lambda_3} = \{0\} \subset \mathbb{Q}[x_1+x_2, x_1x_2].$$

$$\mathrm{Ind}_{\lambda_1} : \mathbb{Q}[x_1, x_2] \longrightarrow \mathbb{Q}[x_1+x_2, x_1x_2]$$
$$f(x_1, x_2) \longmapsto \frac{x_1^g f(x_1,x_2) - x_2^g f(x_2, x_1)}{x_1 - x_2}$$

$$\mathrm{Ind}_{\lambda_2,\lambda_3} : \mathbb{Q}[x_1, x_2] \longrightarrow \mathbb{Q}[x_1+x_2, x_1x_2]$$
$$f(x_1, x_2) \longmapsto \frac{f(x_1, x_2) - f(x_2, x_1)}{x_1 - x_2}$$

surjective $\Rightarrow P_{\lambda_3} = \{0\}$.

Integrality isomorphism

$$P_{\lambda_0} \oplus \left(P_{\lambda_1} \otimes \mathbb{Q}[x_1]\right) \oplus \left(P_{\lambda_2} \otimes \mathbb{Q}[x_1, x_2]\right)^{sgn} \longrightarrow \mathbb{Q}[x_1+x_2, x_1x_2]$$

$$(f, h, k) \mapsto f + \frac{x_1^g h(x_1,x_2) - x_2^g h(x_2,x_1)}{x_1 - x_2} + 2(x_1x_2)^g \frac{k(x_1,x_2)}{x_1-x_2}.$$

<u>exercise</u>: Check by hand this is an iso.

Critical cohomological integrality

Corollary In the same situation as before, for any function $f : X \to \mathbb{C}$,

$$\pi_* \varphi_f \mathbb{Q}^{vir}_{X/G} \cong \bigoplus_{\tilde{\lambda} \in P_X/W} \left[\mathcal{C}_{\lambda *} \left[\varphi_f \mathcal{BPS}_{X/G,\lambda} \otimes H^*(pt/G_\lambda) \right] \right]^{W_\lambda}$$

Dimensional reduction

X weakly symplectic affine algebraic variety

$$TX \underset{\psi}{\cong} T^*X$$

$G \curvearrowright X$ weakly Hamiltonian action :

$\exists \quad \mu : X \to \mathfrak{g}^*$ such that

$$\begin{tikzcd} X \times \mathfrak{g} \arrow[r, "d\mu(x)(\xi)"] \arrow[d, "\varphi"'] & T^*X \arrow[d, "\psi"] \\ X \times \mathfrak{g} \arrow[r] & TX \end{tikzcd}$$

commutes over $\mu^{-1}(0) \subset X$.
subscheme.

Consider $\mu^{-1}(0)/G \longrightarrow \mu^{-1}(0)/\!/G$, related to quiver varieties studied by Nakajima

$f : X \times \mathfrak{g} \longrightarrow \mathbb{C}$
$(x, \xi) \longmapsto \mu(x)(\xi)$.

Dimensional reduction $\quad p : X \times \mathfrak{g} \to X \quad ; \quad p : X \times \mathfrak{g} /\!/ G \to X /\!/ G$

$$p_* \varphi_f \mathbb{Q}^{vir}_{X\times\mathfrak{g}/G} \cong \mathbb{D}\mathbb{Q}^{vir}_{\mu^{-1}(0)/G} .$$

Corollary : The critical coh. integrality isomorphism provides an isomorphism

$$\pi_* \mathbb{D}\mathbb{Q}^{vir}_{\mu^{-1}(0)/G} \cong \bigoplus_{\tilde{\lambda} \in P_X / W} \left[p_* \iota_* \varphi_f \mathcal{BPS}_{V/G,\lambda} \otimes H^*(pt/G_\lambda) \right]^{W_\lambda}.$$

Corollary : purity $X \circlearrowleft G$ weakly Hamiltonian

① $\pi_* \mathbb{D}\mathbb{Q}^{vir}_{\mu^{-1}(0)/G} \in \mathcal{D}^+_G(\mathrm{MMHM}(\mu^{-1}(0)))$ is a pure complex.

② $T^*V \circlearrowleft G$ Hamiltonian action on T^*V; $\mu : T^*V \to \mathfrak{g}^*$ moment map.

$H^{BM}_*(\mu^{-1}(0)/G, \mathbb{Q})$ is pure.

Then $H^{BM}_*(\mu^{-1}(0)/G)$ carries a pure MHS

③. If $\mathfrak{X}$ is a 1-Artin derived stack with affine diagonal, good moduli space $\mathfrak{X} \xrightarrow{\pi} X$ and

X is proper, $H_*^{BM}(\mathfrak{X})$ has pure MHS.
[conjecture of Halpern-Leistner]

① $\Rightarrow$ ③ Any such stack is $\overset{\text{étale}}{\text{locally}}$ of the form

$$\begin{array}{c} \mu^{-1}(0)/G \\ \downarrow \\ \mu^{-1}(0)/\!/G \end{array}$$

and so $\pi_* \mathbb{D}\mathbb{Q}_{\mathfrak{X}}^{vir}$ is a pure complex of MMHM.

Since X is proper, $(X \to pt)_* \pi_* \mathbb{D}\mathbb{Q}_{\mathfrak{X}}^{vir}$ is pure again.

$$\overset{\shortparallel}{H_*^{BM}(\mathfrak{X}, \mathbb{Q})}.$$

① $\Rightarrow$ ② : purity of $\pi_* \mathbb{D}\mathbb{Q}_{\mu^{-1}(0)/G}^{vir} \in \mathcal{D}_G^+(\mathrm{MMHM}(\mu^{-1}(0)))$ and $\mathbb{C}^*$-equivariance.

① interplay between weight properties of $\pi_* \mathbb{D}\mathbb{Q}_{\mu^{-1}(0)}^{vir}$, self-duality properties for $\mathcal{BPS}_{V/G,d}$ and support property for $\mathcal{BPS}_{V/G,d}$. ∎

Corollary : $H^{BM}(\mathrm{Higgs}_G^{sst}(C))$ is pure.

Future directions

* coh. integrality for non-necessarily symmetric stacks or representations : I think yes